Dimension reduction in the context of structured deformations

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Topics in the Calculus of Variations: Recent Advances and New Trends

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Scope - I

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Examples are

(i) finely layered bodies (stack of papers), granular bodies (pile of sand), bodies with defects (metal bar);

(ii) membranes (sheet of rubber), thin plates (sheet of metal), fibered thin bodies (sheet of paper).

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Information about the microstructure can be lost in the dimension reduction procedure.

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Introduced to provide a multiscale geometry that captures the contributions at the macrolevel of both smooth geometrical changes and non-smooth geometrical changes (disarrangements) at submacroscopic levels[3].

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A structured deformation is a pair $(g, G) \in SBV \times L^1$ with $Dg = \nabla g \mathcal{L}^N + [g] \otimes \nu \mathcal{H}^{N-1}$.

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Approximation Theorem: there exists $f_n \in SBV$ such that

$$f_n \stackrel{L^1}{\to} g \qquad \text{and} \qquad \nabla f_n \stackrel{\mathcal{M}}{\rightharpoonup} G.$$

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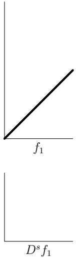

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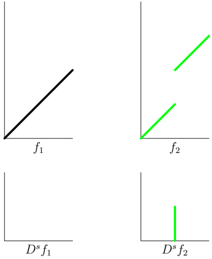

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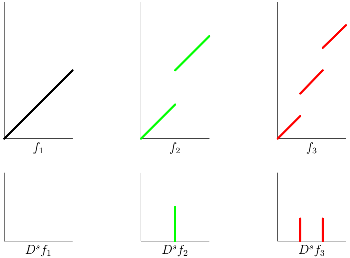

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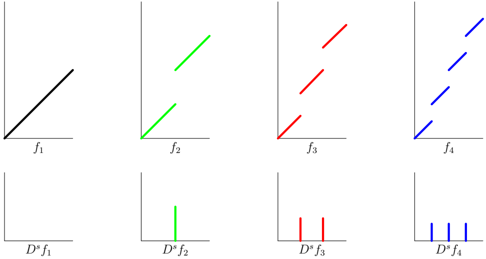

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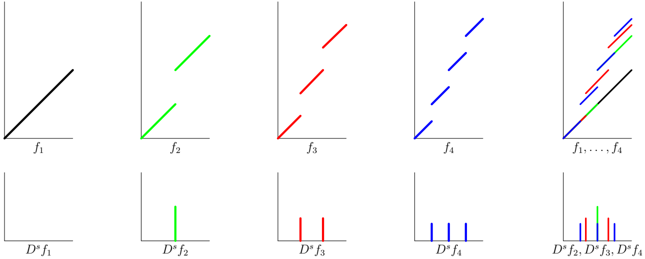

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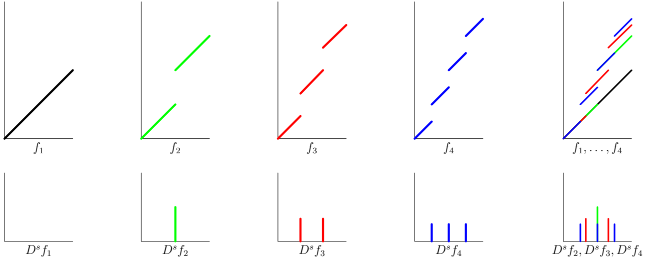

- *deck of cards*: $N = 3$, $\Omega = (0,1)^3$, $\kappa = \emptyset$, $g(x) = (x_1 + x_3, x_2, x_3)$, and $G(x) = \mathbb{I}$.

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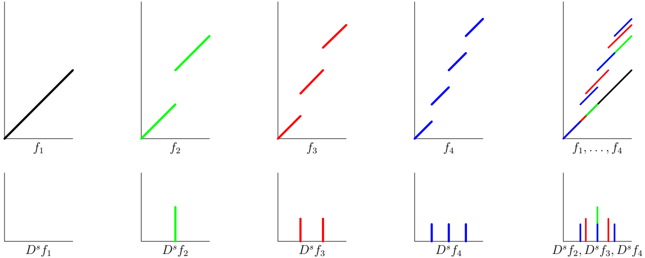

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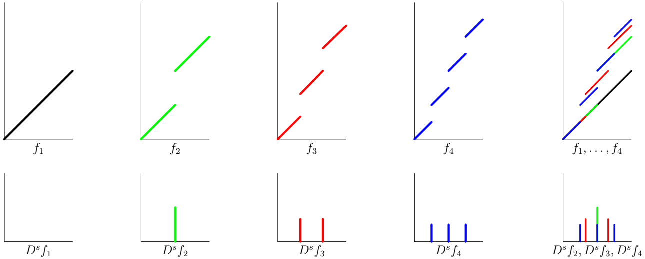

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More on Disarrangements

From the examples it should be clear (and this can be formalized) that the singular part $D^s f_n$ (supported on the jump set $S(f_n)$) *diffuses* in the limit to generate volume energy (supported on the bulk).

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Singularities are essentially captured by M and its derivatives. If G and M provide information about plastic deformations, M and $\mathrm{curl}\, M$ allow to describe the *Burgers vectors* and the *dislocation* density field in a body containing defects.

So, $M = \nabla g - G$ is a measurement of how non classical a deformation is.

Energies

Typical energies of interest in this context are of the form

$$E(u) = \int_{\Omega} W(\nabla u, \nabla^2 u) + \int_{S(u)} \psi_1([u], \nu_u) + \int_{S(\nabla u)} \psi_2([\nabla u], \nu_{\nabla u}),$$

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- W depending on A includes bending effects.

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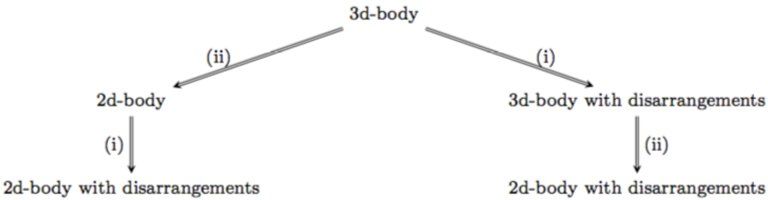
3d-body
(ii)
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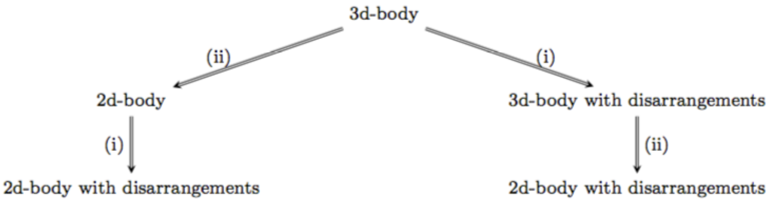

Programme: do the two relaxation procedures and find an integral representation.

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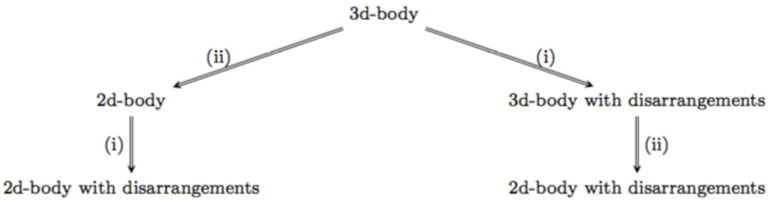

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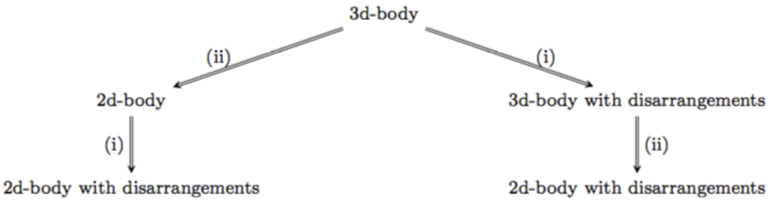

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(b) Does a simultaneous relaxation procedure yield a lower energy (what about a central path)?

Relaxation

Relaxing the energy E means to compute

$$I(g, G, \Gamma) := \inf_{\{u_n\} \subset SBV^2} \left\{ \liminf_{n \to \infty} E(u_n) : u_n \xrightarrow{L^1} g, \nabla u_n \xrightarrow{L^1} G, \nabla^2 u_n \overset{*}{\rightharpoonup} \Gamma \right\}$$

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and possibly to get a *representation formula*,where the bulk and surface densities are obtained by a *cell formula*,[5] derived by a *blow-up method*.[6]

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In the formula above, we are looking for the most economical way to approximate the (second-order)[7] structured deformation (g,G,Γ) by means of more regular deformations.

---

[5] Choksi, Fonseca – ARMA (1997)

[6] Fonseca, Müller – SIAM J. Math. Anal. (1992)

[7] Barroso, Matias, M., Owen – ARMA (2017)

Relaxation *à la* Choksi-Fonseca - I

The relaxation of an energy like

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leads to the *representation formula*

$$I(g,G) = \int_\Omega H(\nabla g, G)\,\mathrm{d}\mathcal{L}^N + \int_{S(g)\cap\Omega} h([g],\nu(g))\,\mathrm{d}\mathcal{H}^{N-1}.$$

Relaxation *à la* Choksi-Fonseca - II

The densities H and h are given by

$$H(A,B) := \inf\Big\{ \int_Q W(\nabla u)\,\mathrm{d}\mathcal{L}^N + \int_{S(u)\cap Q} \psi([u],\nu(u))\,\mathrm{d}\mathcal{H}^{N-1} :$$
$$u \in SBV(Q;\mathbb{R}^N), u_{|\partial Q}(x) = Ax,\ |\nabla u| \in L^p(Q),\ \int_Q \nabla u = B \Big\},$$

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where

$$u_{\xi,\eta}(x) := \begin{cases} \xi & \text{if } 0 \leqslant x\cdot\eta < 1/2, \\ 0 & \text{if } -1/2 < x\cdot\eta < 0. \end{cases}$$

Dimension reduction in the context of SD[8]

$$E_\varepsilon(u) := \int_{\Omega_\varepsilon} W_{3d}(\nabla u)\,\mathrm{d}x + \int_{\Omega_\varepsilon \cap S(u)} h_{3d}\big([u], \nu(u)\big)\,\mathrm{d}\mathcal{H}^2$$

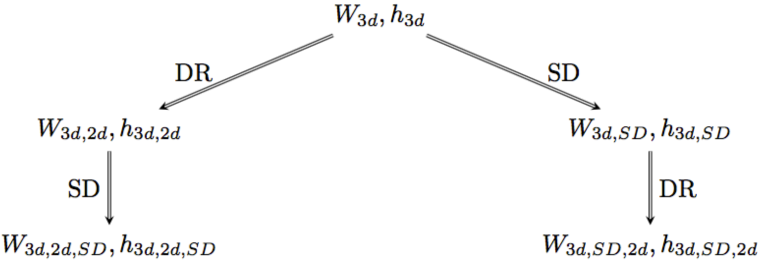
for $u \in SBV(\Omega_\varepsilon;\mathbb{R}^3)$, with $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

[8] Carita, Matias, M., Owen – J. Elast. (2018)

Dimension reduction in the context of SD[8]

$$E_\varepsilon(u) := \int_{\Omega_\varepsilon} W_{3d}(\nabla u)\,\mathrm{d}x + \int_{\Omega_\varepsilon \cap S(u)} h_{3d}([u], \nu(u))\,\mathrm{d}\mathcal{H}^2$$

for $u \in SBV(\Omega_\varepsilon; \mathbb{R}^3)$, with $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.



Left-hand side: first dim. red., then structured deformations;
Right-hand side: first structured deformations, then dim. red.

[8] Carita, Matias, M., Owen – J. Elast. (2018)

Hypotheses on the energy densities

We assume that:

(H_1) There exists a constant $c_W > 0$ such that growth conditions from above and below are satisfied

$$\frac{1}{c_W}|A|^p \leqslant W_{3d}(A),$$

$$|W_{3d}(A) - W_{3d}(B)| \leqslant c_W|A - B|(1 + |A|^{p-1} + |B|^{p-1}),$$

for any $A, B \in \mathbb{R}^{3\times 3}$, and for some $p > 1$.

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(H_2) There exists a constant $c_h > 0$, such that for all $(\lambda, \nu) \in \mathbb{R}^3 \times \mathbb{S}^2$

$$\frac{1}{c_h}|\lambda| \leqslant h_{3d}(\lambda, \nu) \leqslant c_h|\lambda|.$$

(H_3) $h_{3d}(\cdot, \nu)$ is *positively 1-homogeneous*: for all $t > 0$, $\lambda \in \mathbb{R}^3$

$$h_{3d}(t\lambda, \nu) = t\, h_{3d}(\lambda, \nu).$$

(H_4) $h_{3d}(\cdot, \nu)$ is *subadditive*: for all $\lambda_1, \lambda_2 \in \mathbb{R}^3$

$$h_{3d}(\lambda_1 + \lambda_2, \nu) \leqslant h_{3d}(\lambda_1, \nu) + h_{3d}(\lambda_2, \nu).$$

Dimension reduction

Rescale by ε in x_3 and consider the functional $F_\varepsilon(u)$

$$\frac{E_\varepsilon(u)}{\varepsilon} = \int_\Omega W_{3d}\left(\nabla_\alpha u \middle| \frac{\nabla_3 u}{\varepsilon}\right) \mathrm{d}x + \int_{\Omega\cap S(u)} h_{3d}\left([u], \nu_\alpha(u) \middle| \frac{\nu_3(u)}{\varepsilon}\right) \mathrm{d}\mathcal{H}^2(x).$$

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The coercivity assumption grants boundedness of the gradients in L^p, so that $\varepsilon_n^{-1}\nabla_3 u_n$ has a weak limit $d \in L^p(\Omega;\mathbb{R}^3)$. Therefore, given $(\overline{u},\overline{d}) \in SBV(\omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$, let

$$\mathcal{F}_{3d,2d}(\overline{u},\overline{d}) := \inf\left\{ \liminf_{n\to\infty} F_{\varepsilon_n}(u_n) : u_n \in SBV(\Omega;\mathbb{R}^3), u_n \xrightarrow{L^1(\Omega;\mathbb{R}^3)} \overline{u}, \int_I \frac{\nabla_3 u_n}{\varepsilon_n}\,\mathrm{d}x_3 \rightharpoonup \overline{d} \text{ in } L^p(\omega;\mathbb{R}^3),\ \nu(u_n)\cdot e_3 = 0 \right\}.$$

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Theorem (Carita-Matias-M.-Owen (2018))

$$\mathcal{F}_{3d,2d}(\overline{u},\overline{d}) = \int_\omega W_{3d,2d}(\nabla\overline{u},\overline{d})\,\mathrm{d}x_\alpha + \int_{\omega\cap S(\overline{u})} h_{3d,2d}([\overline{u}],\nu(\overline{u}))\,\mathrm{d}\mathcal{H}^1(x_\alpha).$$

Integral representation

Theorem (Carita-Matias-M.-Owen (2018) – cont'd)

$W_{3d,2d}\colon \mathbb{R}^{3\times 2}\times\mathbb{R}^3\to[0,+\infty)$ *and* $h_{3d,2d}\colon \mathbb{R}^3\times\mathbb{S}^1\to[0,+\infty)$ *are*

$$W_{3d,2d}(A,d)=\inf\bigg\{\int_{Q'} W_{3d}(\nabla_\alpha u|z)\,\mathrm{d}x_\alpha+\int_{Q'\cap S(u)} h_{3d}([u],\tilde{\nu}(u))\,\mathrm{d}\mathcal{H}^1(x_\alpha):$$
$$u\in SBV(Q';\mathbb{R}^3),\ z\in L^p_{Q'-\mathrm{per}}(\mathbb{R}^2;\mathbb{R}^3),\ u|_{\partial Q'}(x_\alpha)=Ax_\alpha,\int_{Q'} z\,\mathrm{d}x_\alpha=d\bigg\},$$

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$$h_{3d,2d}(\lambda,\eta) = \inf\Big\{ \int_{Q'_\eta\cap S(u)} h_{3d}([u],\tilde\nu(u))\,\mathrm{d}\mathcal{H}^1(x_\alpha) : u \in SBV(Q'_\eta;\mathbb{R}^3), \\ u|_{\partial Q'_\eta}(x_\alpha) = \gamma_{\lambda,\eta}(x_\alpha),\ \nabla u = 0,\ \textit{a.e.} \Big\};$$

with

$$\gamma_{\lambda,\eta}(x_\alpha) := \begin{cases} \lambda & \textit{if } 0 \leqslant x_\alpha\cdot\eta < \frac12, \\ 0 & \textit{if } -\frac12 < x_\alpha\cdot\eta < 0. \end{cases}$$

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- lower bounds for the Radon-Nikodým derivatives of μ, the weak-* limit of the measures μ_n

$$\mu_n(B) := \int_{B\times I} W_{3d}\Big(\nabla_\alpha u_n \Big| \frac{\nabla_3 u_n}{\varepsilon_n}\Big)\mathrm{d}x + \int_{(B\times I)\cap S(u_n)} h_{3d}([u_n], \tilde{\nu}(u_n))\mathrm{d}\mathcal{H}^2(x).$$

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The doubly relaxed energies

Theorem (Carita-Matias-M.-Owen (2018))

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Recall that

$$\frac{1}{\varepsilon_n}\int_I \nabla_3\, u_n\,\mathrm{d}x_3 \overset{L^p}{\rightharpoonup} \overline{d}\,:$$

the vector $\overline{d}$ emerges as the weak limit of the out-of-plane deformation gradient.

An example

Consider an initial energy $\boldsymbol{E}_\varepsilon$ in which the densities are $W_{3d} = 0$ and $h_{3d}(\lambda, \nu) = |\lambda \cdot \nu|$.

Theorem (Carita-Matias-M.-Owen (2018))

Let $W_{3d} = 0$ and $h_{3d}(\lambda, \nu) = |\lambda \cdot \nu|$. Then the two functionals $\mathcal{F}_{3d,2d,SD}$ and $\mathcal{F}_{3d,SD,2d}$ coincide (and neither one depends on $\overline{d}$):

$$\mathcal{F}_{3d,SD,2d}(\overline{g}, \overline{G}, \overline{d}) = \widehat{\mathcal{F}}_{3d,SD,2d}(\overline{g}, \overline{G}) = \int_\omega |\operatorname{tr}(\widehat{\nabla \overline{g}} - \widehat{\overline{G}})| \, \mathrm{d}x_\alpha + \int_{\omega \cap S(\overline{g})} |[\overline{g}] \cdot \tilde{\nu}(\overline{g})| \, \mathrm{d}\mathcal{H}^1(x_\alpha).$$

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The result is in agreement with previous results in the literature.[9]

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[9] Owen, Paroni – ARMA (2015)
Barroso, Matias, M., Owen – MEMOCS (2017)
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Comparison with other relaxation procedures

For a function $u \in SBV^2(\Omega_\varepsilon;\mathbb{R}^3)$, consider the initial energy[10]

$$E_\varepsilon^{MS}(u) := \int_{\Omega_\varepsilon} W(\nabla u, \nabla^2 u)\,\mathrm{d}x + \int_{\Omega_\varepsilon\cap S(u)} \Psi_1([u],\nu(u))\,\mathrm{d}\mathcal{H}^2(x) + \int_{\Omega_\varepsilon\cap S(\nabla u)} \Psi_2([\nabla u],\nu(\nabla u))\,\mathrm{d}\mathcal{H}^2(x)$$

[10] Matias, Santos – Appl. Math. Optim. (2014)

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and the relaxation of the rescaled energy $J_\varepsilon(u) := \frac{1}{\varepsilon}E^{MS}_\varepsilon(u)$

$$I(g,G,d) := \inf\Big\{\liminf_{n\to\infty} J_{\varepsilon_n}(u_n) : u_n \in SBV^2(\Omega;\mathbb{R}^3),\, u_n \stackrel{L^1}{\to} g,\, \frac{1}{\varepsilon_n}\nabla_3 u_n \stackrel{L^1}{\to} d,\, \nabla_\alpha u_n \stackrel{L^1}{\to} G\Big\},$$

Theorem (Carita-Matias-M.-Owen (2018))

The simultaneous procedure yields a relaxed energy which is lower than the two sequential procedures.

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Theorem (Carita-Matias-M.-Owen (2018))

The simultaneous procedure yields a relaxed energy which is lower than the two sequential procedures.
In fact, in the case $W_{3d} = 0$ *and* $h_{3d}(\lambda,\nu) = |\lambda\cdot\nu|$*, the relaxed energy is always equal to zero.*

[10] Matias, Santos – Appl. Math. Optim. (2014)

The functional I admits an integral representation $I = I_1 + I_2$, where, for $(g, G) \in BV^2(\omega; \mathbb{R}^3) \times BV(\omega; \mathbb{R}^{3\times 2})$,

$$I_1(g, G) = \int_\omega W_1(G - \nabla g)\, \mathrm{d}x_\alpha + \int_\omega W_1\Big(-\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\Big)\, \mathrm{d}|D^c g|(x_\alpha) + \int_{\omega\cap S(g)} \Gamma_1([g], \nu(g))\, \mathrm{d}\mathcal{H}^1(x_\alpha)$$

and for $(d, G) \in BV(\omega; \mathbb{R}^3) \times BV(\omega; \mathbb{R}^{3\times 2})$

$$I_2(d, G) = \int_\omega W_2(d, G, \nabla d, \nabla G)\, \mathrm{d}x_\alpha + \int_\omega W_2^\infty\Big(d, G, \frac{\mathrm{d}D^c(d, G)}{\mathrm{d}|D^c(d, G)|}\Big)\, \mathrm{d}|D^c(d, G)| + \int_{\omega\cap S((d,G))} \Gamma_2((d, G)^+, (d, G)^-, \nu((d, G)))\, \mathrm{d}\mathcal{H}^1(x_\alpha)$$

The energy densities of I_1 are obtained as follows: for each $A \in \mathbb{R}^{3\times 2}$, $\lambda \in \mathbb{R}^3$, and $\eta \in \mathbb{S}^1$,

$$W_1(A) = \inf\Big\{\int_{Q'\cap S(u)} \overline{\Psi}_1([u], \nu(u))\, \mathrm{d}\mathcal{H}^1(x_\alpha) : u \in SBV(Q'; \mathbb{R}^3), u|_{\partial Q'} = 0, \nabla u = A \ a.e.\Big\},$$

$$\Gamma_1(\lambda, \eta) = \inf\Big\{\int_{Q'_\eta\cap S(u)} \overline{\Psi}_1([u], \nu(u))\, \mathrm{d}\mathcal{H}^1(x_\alpha) : u \in SBV(Q'_\eta; \mathbb{R}^3), u|_{\partial Q'_\eta} = \gamma_{\lambda,\eta}, \nabla u = 0 \ a.e.\Big\},$$

with $\overline{\Psi}_1(\lambda, \nu) := \inf\{\Psi_1(\lambda, (\nu|t)) : t \in \mathbb{R}\}$. For each $A \in \mathbb{R}^{3\times 2}$, $B_\beta \in \mathbb{R}^{3\times 3\times 2}$, $\Lambda, \Theta \in \mathbb{R}^{3\times 3\times 2}$, and $\eta \in \mathbb{S}^1$,

$$W_2(A, B_\beta) = \inf\Big\{\int_{Q'} \overline{W}(A, \nabla u)\, \mathrm{d}x_\alpha + \int_{Q'\cap S(u)} \overline{\Psi}_2([u], \nu(u))\, \mathrm{d}\mathcal{H}^1(x_\alpha) : u \in SBV(Q'; \mathbb{R}^{3\times 3}), u_{ik}|_{\partial Q'} = \sum_{j=1}^{2} B_{ijk}x_j\Big\},$$

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where

$$u_{\Lambda,\Theta,\eta}(x_\alpha) := \begin{cases} \Lambda & \text{if } 0 \leqslant x_\alpha \cdot \eta < 1/2, \\ \Theta & \text{if } -1/2 < x_\alpha \cdot \eta < 0, \end{cases}$$

and with $\overline{W}$ and $\overline{\Psi}_2$ as follows: decomposing $B \in \mathbb{R}^{3\times 3\times 3}$ into $(B_\beta, B_3) \in \mathbb{R}^{3\times 3\times 2} \times \mathbb{R}^{3\times 3\times 1}$ (i.e., B_β denotes B_{ijk} with $k = 1, 2$), define $\overline{W}(A, B_\beta) := \inf\{W(A, (B_\beta, B_3)) : B_3 \in \mathbb{R}^{3\times 3\times 1}\}$, and for $\Lambda \in \mathbb{R}^{3\times 3}$ and $\eta \in \mathbb{S}^1$, let $\overline{\Psi}_2(\Lambda, \eta) := \inf\{\Psi_2(\Lambda, (\eta|t)) : t \in \mathbb{R}\}$.

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Thank you for your attention!

[11] Matias-M.-Owen-Zappale – *in progress*

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